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Article

In-service Teachers Mental Construction of Linear Algebra Concepts. An Undergraduate Case Study

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Abstract: This study investigated how undergraduate in-service mathematics teachers understand linear algebra concepts. The significance of the study is to reveal the difficulties and type of mental constructions they make when they are solving linear independence concepts by using inspection. The action, process, object, schema (APOS) theory was used to analyse the data collected. A descriptive qualitative case study design was used. The participants were 73 in-service teachers studying a Bachelor of Science Education Honours Degree in mathematics. Data was collected through a structured activity sheet and semi-structured interviews. The results showed that the teachers had an inadequate understanding of linear independence concepts. Many of the students developed the action conceptions of linear algebra as they were engrossed in step-by-step procedures trying to show that given vectors are linearly independent. The findings indicated that the schemata for determinants and the various theorems for determining linear independence promote learning. The modified genetic decomposition proposed has implications for teaching the concept and students should be given different kinds of sets to manipulate so that they can develop their mental constructions at the process level.

Keywords: APOS theorem, linear algebra, linear independence, in-service teachers, understanding

Introduction

Bouhjar et al. (2018) outlines that linear algebra is a compulsory course and it is widely recognised to have important applications in engineering, science, technology, and mathematics. Klapsou and Gray (1999) and Tucker (1993) further asserted that it is the course that students first experience at university level. However, so many researchers were worried about the poor achievements students experience when learning matrix algebra (Stewart & Thomas, 2010; Ndlovu & Briglall, 2015; Kazunga & Bansilal, 2020; Mutambara and Bansilal, 2022; Wawro, Sweeney & Rabin, 2011; Maharaj, 2015). There is a consensus that this course is problematic for the students. Dorier and Sierpinska (2001) posited that "the nature of linear algebra (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)" were the foundation of the challenges students have with the learning of linear algebra course. On another dimension Dogan-Dunlap (2010) posited that too much formalisation plays a greater part in causing these challenges that students encounter. The students have a feeling that the work they are given is not in line with their prior knowledge. There is the need for these learners to embark on new terms, symbols and the need to define these new terms and propositions, (Dorier et al. 2000).

In their studies, Stewart and Thomas (2009) and Carlson (1997) were of the opinion that students were able to solve problems involving matrices but encounter problems with concepts such as subspaces and linear independence. Carlson (1997) also noted that students struggle to understand concepts in linear algebra. The researcher in this study has taught linear algebra at advanced level as well as at the university and have also taken note that students struggle to understand the vector space concepts. It is against the learning difficulties that drove the researcher to embark on such a study. The aim of the study was to reveal the nature of mental constructions made by the in-service teachers when learning linear independence concepts by using inspection and its contribution to instructional strategies. The following objectives were formulated:

- To identify the mental constructions the students have developed when learning linear independence concepts.
- To investigate the challenges that students face when learning linear independence concepts using theorems.

It is hoped that the identification of difficulties that students encounter when solving linear independence involving theorems will enlighten the concerned instructors about what to be aware of as they teach the concepts.

Review of Literature

At various universities, the first module that the students encounter is on linear algebra. Researchers such as Mutambara and Bansilal (2019) noted that students struggle to understand these concepts because they have a tendency of memorising definitions from different sources thus hampering them the ability to conceptualise these concepts. Wawro, Sweeney and Rabin (2011) sees reasoning as an important part in the learning of mathematics. These researchers further purported that for one to attain such a skill, they must be able to engage constructively when defining mathematical concepts, be able to do proofs successfully and making correct arguments with proper justification. This implies that there is the need to carry out research with the in-service teachers because they need to teach these concepts at advanced level and thus it has suggestions for classroom practice.

In this section we summarise the research findings on the concepts on linear algebra. In recent studies Ndlovu and Brijlall (2015) used the APOS theory to see how students understand concepts in matrix algebra. The researchers discovered that many of the students established their mental construction at the action level, according to the APOS theory. Stewart and Thomas (2010) used the APOS theory and Tall (2004) theory of three worlds of mathematics to scrutinise the students' understanding of linear independence and spanning. The author's designed a genetic decomposition of the concepts before teaching the students. The study involved ten students. The three worlds of mathematics according to Tall (2004) encompasses the embodied, symbolic and the formal world. The findings of the study showed that students totally failed to interpret linear independence geometrically as well as defining the stated terms. They further argued that the abstract and the formal nature of these concepts caused the students to encounter these difficulties.

Another study that used APOS theory was carried by Kazunga and Bansilal (2017). The theory was used to assess how undergraduate students understand concepts in matrices. Students were given some written work followed by semi structured interviews and it was noted that students had challenges in multiplying the matrices whose order is different for example matrices of order $m \times 1$ and $1 \times$ m. One of the participants attempted to simplify$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$. These matrices are of different orders and the participant obtained $(2 \times 1 + 0 \times 2 + 1 \times (-1)) = 1$. The student struggled to simplify the expression, and this shows that the concept of multiplication of matrices with different orders was not mastered.

The definition of linear independence is stated as follows: If S = $\{v_1, v_2, \dots v_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1 v_1 + k_2 v_2 + \cdots k_n v_r = 0$ has at least one solution namely $k_1 = 0,\ k_2 = 0, \dots k_r = 0$. The results of these studies inspired us to use the APOS theory to see how the in-service mathematics teachers construct their knowledge when studying the concept of linear independence using theorems.

Theoretical framework

Dubinsky and Macdonald (2001) proposed the APOS theory and this study is underpinned by this theorem. Ndlovu and Brijlall (2015) outlined that the theory has been extended to understand students' mental constructions at secondary and tertiary level. It points towards pedagogical strategies that lead to marked changes in students' learning of complex or abstract mathematical concepts.

Arnon, et al. (2014) further pointed out that the ideas of this theory are based on Piaget's (1973) belief that an individual has the ability to learn mathematics by relating certain mental mechanisms to build exact mental structures. The individual arrives at new knowledge by reflecting on prior knowledge without the need for information added to it. He goes on to say the theory is useful when students build concepts in subjects like statistics, discrete mathematics, calculus and abstract algebra. APOS theory is the acronym for Action, Process, Object and Schema. The APOS theory is going to be defined with the help of Dubinsky and McDonald (2001) ideas.

Action: This involves the step by step procedures as a result of a reaction to a stimulus which an individual perceives as external and this depends on specific rules.

Process: A process is observed as being internal and takes place in the mind of an individual without step by step procedures.

Object: Here an individual imitates on operations applied to a process and become aware of the process as a totality

Schema: Many actions, processes and objects are interconnected in the individual's mind and these will be organized to form a coherent framework called a schema.

In order to apply the APOS theory, one needs to develop a theoretical model called the genetic decomposition. Ndlovu and Brijlall (2015) asserted that a genetic decomposition describes how a concept can develop in one's mind. Possani et al. *(*2010) commented that in order to understand a mathematical concept, an individual construct the exact mental construction of action, process and object. The model is designed based on the researchers' experiences of a given concept so as to improve classroom practice. The proposed genetic decomposition for the concept of linear independence is presented below.

The genetic decomposition for linear independence

Action: In order to show that a given set of vectors is linearly independent, the individual has to formulate equation of the form $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$, and the stage is externally driven. The method of elementary row operation or the determinant is used.

Process: Here the individual makes use of arguments based on specific theorems to show linear independence without following a specific algorithm.

Object level: In the case of linear independence the individual is able to use the method of row reduction or finding the determinant and make informed decision based on solution obtained.

Methodology

1. Research Design

A descriptive qualitative case study design was used. According to Denzin and Lincoln (2011) a qualitative research accounts for the researchers understanding of participants social and cultural occurrences. The researcher relies on the views of the participants in order to understand their cultural phenomena, Creswell (2014). Bertram and Christiansen (2014) also acknowledged the benefits of a case study and said that it leads to an in-depth description of data collected and unearth issues that require further investigation. The researcher employed semi structured interviews and a written activity sheet with questions so as get deeper insight of the problem at hand. However, the major limitation of the interview as a method of data collection is that it provides less anonymity such that some respondents might not open up.

2. Sampling

The study used purposive sampling, where 73 students studying for the Bachelor of Science Honours Degree in mathematics volunteered to participate. The researcher selected the cases that are accessible and have in-depth knowledge of the concepts under study. These students specialised in mathematics and the programme was done over a duration of three years. These students were already teaching mathematics at ordinary level, and they have graduated with a Diploma in Education in mathematics attained from the various teachers' colleges in Zimbabwe. These teachers were upgrading their qualifications so that they can teach mathematics at Advanced level. The course was offered as a block release during the school holidays.

3. Research Instruments

The research instrument comprised of one question with five tasks. A structured activity sheet was used to collect the data, and concepts on linear independence were set. The tasks were incorporated using the genetic decomposition. Semi structured interviews were also conducted. The interviews enabled the researcher to gather information which was too complex to be gathered by written responses only.

4. Research Procedures

In order to collect the data, the in-service teachers were first taught by the researcher. Students answered questions from the activity individually and being supervised by the researcher. After marking the work, some follow up interviews were accompanied with nine participants basing on their responses. To ensure confidentiality and anonymity of the participants, pseudonyms were used and the students were coded using tags 'T1', 'T2' up to T73 but the order did not have any meaning. The researcher collected the data and later on draw important information and made inferences based on the patterns or themes emerging. The semi structured interviews were conducted where the pre-set questions left room for further questions in order to solicit more information. An in-depth content analysis of the 73 students written responses was done. The areas that students encountered difficulties were noted. The analysis was done identifying emerging patterns within the data using APOS theory thus identifying the mental constructions that the students revealed basing on action and process reasoning and misconceptions that the students have from the written work. The percentages of students getting the correct result was noted.

The Findings

The structured activity sheet had a question with five-part questions. According to the genetic decomposition, these questions tested the process level of understanding of linear independence. The question is shown in table 1, together with percentages with correct result for each question.

Table 1**.** Items and percentages of students with correct results

1. Determine whether the given vectors are linearly independent or not with justification. (Solve the problem by inspection).
   (a) $\boldsymbol{u_1} = (-1{,}2{,}4)$ and $\boldsymbol{u_2} = (5, -10, -20)$ in R^3.
   (b) $\boldsymbol{u_1} = (3, -1), \boldsymbol{u_2} = (4{,}5)$ and $\boldsymbol{u_3} = (-4{,}7)$ in R^2.
   (c) $p_1 = 3 - 2x + x^2,\ p_2 = (6 - 4x + 2x^2)$ in p_2.
   (d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$.
   (e) $(2, -3{,}7)$, (0,0,0), (3, -1, -4)

Item	**Percentage with correct result**
a	42 (57%)
b	13 (17%)
c	54 (74%)
d	35 (48%)
e	28 (39%)

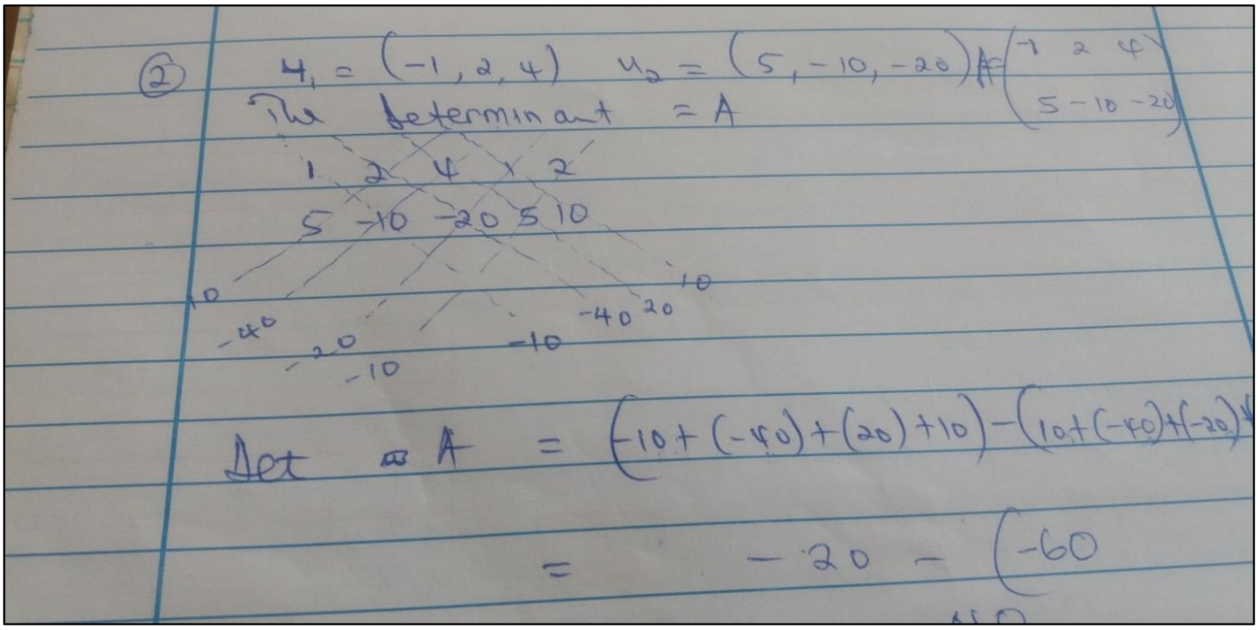
The researcher explored the students understanding of these concepts listed in table 1 to see if they can make informed decisions without using multiple step algorithms. According to the genetic decomposition, all the items tested the process level understanding. This alludes to the attention given by Hannah et al. (2016) that this involves the mental constructions without the need to carry out a calculation.

Considering items a, c, and d there was no need to show any working but required explanations using theorem [1.1] which says " A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other" The results pointed out some weaknesses which included (a) the students attempted to set vector equations, come up with a matrix and carry out Gauss elimination (b) use of the determinant method was prominent, but incorrectly done c) theorems were wrongly applied. Student performed better in item a and c than in item d. The results of the different items are discussed below.

1. Results for Item a

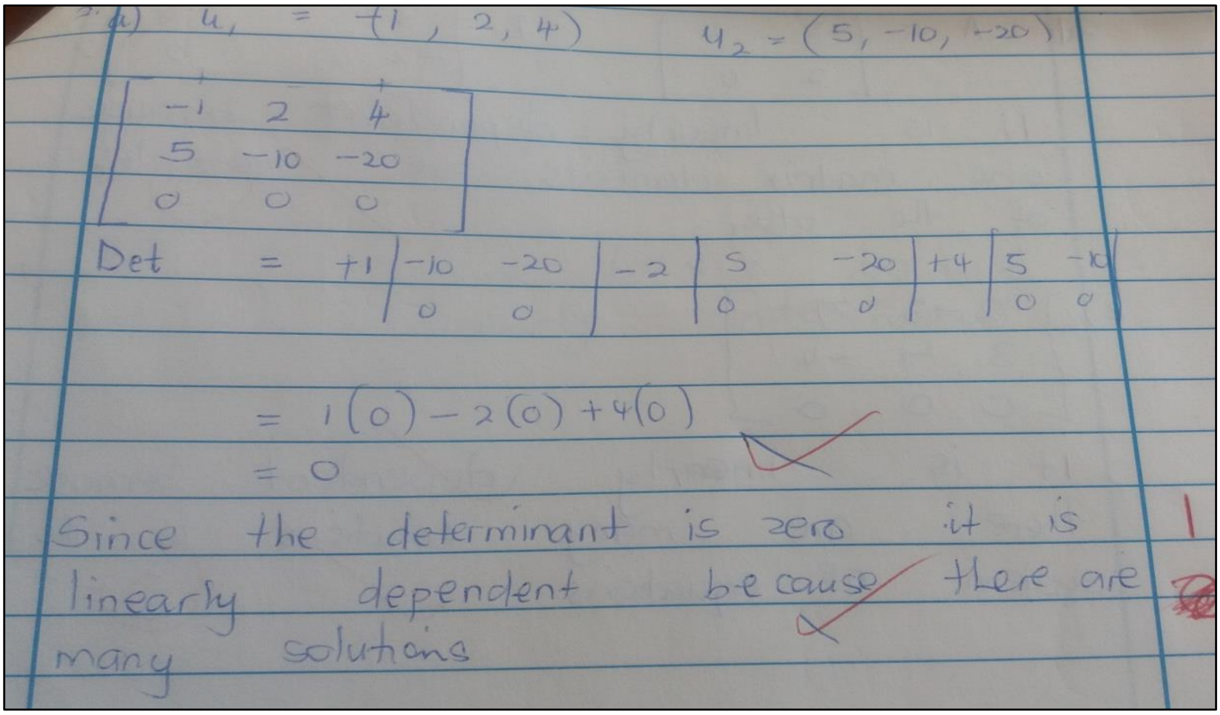
Student T27 constructed a 2× 3 matrix and denoted it matrix A, as shown in Figure 1

Figure 1. Response of T27 using the determinant method wrongly



Student T27 calculated the determinant of matrix A. The matrix was not a square matrix, since it is a 2× 3 matrix. The student used a wrong method to calculate the determinant, and this was used out of the domain. The method used is only valid to a 3×3 matrix. This showed that the students did not master the schema of determinants which was done in the first module. Tomito (2008) commended that the students' prior knowledge has a great impact when learning new material. In this case the determinant method adversely affected the construction of new knowledge. Another Scholar Donevska-Todorova (2016) highlighted that calculating the determinant is regarded as procedural understanding, but if one intends to link the determinant and the inverse then such understanding is referred to as conceptual understanding. In this study students struggled to make a link between the determinant of given vectors and the idea of linear independence. Another student T70 also used the determinant method as shown in figure 2.

Figure 2. Response of T70 forcing a 2× 3 matrix to be an $n \times n$ matrix



The researcher noted that some of the students noticed that the matrix was not square. They added another row of zeros so that it is a square matrix see Figure 2, written response by student T70. The results revealed that the students are well versed with the algorithm that is linked to the determinant method in order to prove linear independence. This prevented them from developing a proper understanding of the concept but developed an action understanding.

The wrong usage of theorems was prominent for example the researcher noted that student T39 wrote that "it is linearly independent because the number of components in vectors are less than the number of vectors since $r > n$, with r columns and n rows". The student failed to remember this theorem such that they ended up grabbing different terms that they may think of such as saying $r < n$ and make wrong conclusions. This shows that all these students are not operating at the process level of understanding.

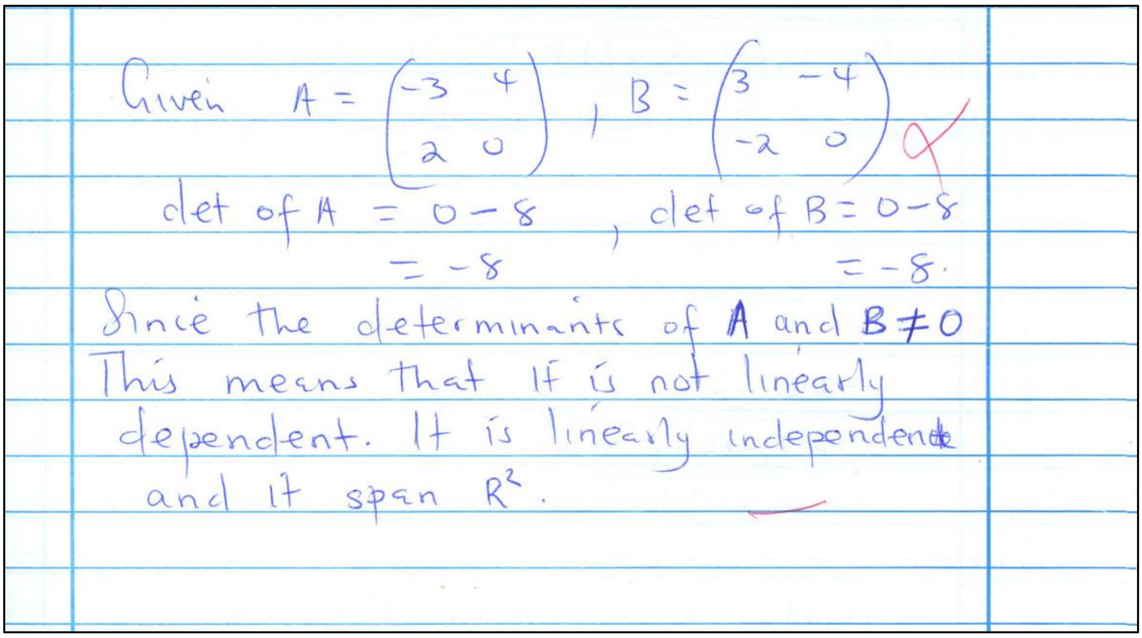
2. Results for item c

The researcher noted that the student did well. 74% of the students obtained the correct solutions. The aspect on scalar multiplication was well represented. Though some of the students were able to identify this property, they could not give the correct argument hence the conclusion was not correct. An example is student T48 who commented that it is linearly dependent since $r < n$. Most of the students reversed the inequality sign for it was supposed to be $r > n$. This hindered the students to make the appropriate mental constructions according to the genetic decomposition.

3. Results for item d

Here the vectors were given in matrix form. Students struggled to prove linear independence. Students continued grabbing any terms inorder to determine linear independence and these were wrongly used. Another student used the wrong method, see figure 3.

Figure 3. Response of T72 showing wrong method



It is interesting to note that some resorted to finding the $det\ A = -8$ and the det $B = -8$ and made decision based on the solution but it was not correct see figure 3, solution of T72.The student's response is incorrect. These students always think of the determinant when proving linear independence. The moment they get $det\ \neq 0$ or $det = 0$ then they can argue whether linearly independent or dependent, without coming to terms with the process involved. This result indicated a failure to interiorise actions into a process of the concept. Some students attempted to find the vector equation of the form $k_1v_1 + k_2v_2 = 0$, and got solutions of the form $k_1 = 0, k_2 = 0$ without showing working. Hannah et al. (2016) highlighted that students are more concerned with the definition and they fail to conceptualise the concept. They further outlined that the students can remember the equation and the testable conditions where say $k_1 = k_2 = 0 = \cdots k_n = 0$. Also solutions of the form A= $-B$ were prominent and the students simply made wrong deductions. This indicates the development of the action stage. The summaries of the results to items a, c and d are displayed in Table 2.

Table 2. Summary of the scores for item a, c and d.

Categories		**Frequency**		
		Part a	Part c	Part d
Wrong solutions	Unreasonable deductions made	16	19	10
	Used the determinant method	6		9
	Used row reduction method	9		7
	Obtained the expression $A = -B$ with wrong deduction			12
Correct responses	Used inspection correctly	42	54	35

From the table, it is seen that many of the students resorted to the use of the different methods with step by step procedures instead of simply using theorems.

4. The semi structured interviews to item a, c, d

Basing on the solutions, students were asked to use their own words in describing how they can test for linear independence when given a set of vectors. With reference to item 1a, T57 gave reference to number of vectors in a vector space which was incorrect. Student T13 had an idea that he needs to talk about one vector being a multiple of the other but was illogically concluded. The researcher presented the student responses to the interview questions:

R*: Directing the students to item a, can you explain whether the following vectors are linearly independent or not?*
T27*: I need to write the vector equation hmmm and come up with systems of equations and then calculate the determinant of the matrix obtained.*
R*: [Probing further] What is the order of the matrix, and how would we calculate the determinant. [trying to draw attention to the fact that W is not a square matrix]*
T27*: [writing down the matrices, and attempting again to use Sarrus rule], the order of the matrix is* 2×3.
R*: Is it possible to calculate the determinant? [probing again about the size of the matrix]*
T27*: Yes can find the determinant of any matrix, isn't it by using Sarrus rule and the method of cofactors.*
(Informant T27, Female, 31 years old)

The researcher noted that the students were still clinking to their misconceptions that in order to determine linear independence, there is the need for the step by step procedures always. This shows that the students had not made the correct mental construction of the concept. This discussion informed the researcher about the crucial role of the determinant in the learning of linear independence.

5. Results for item b

This problem was based on a theorem which says "given a set S = $\{v_1, v_2, \dots v_r\}$ in a real vector, and given $r > n$, then the given set S is linearly dependent", theorem [1.2]. This question was more challenging such that only 18% of the students obtained the correct result. Many of the students could not make proper link considering the unknowns and the equations. Some of the weaknesses that was popular in students' responses was the use of the determinant method. The use of wrong theorems and or wrong justifications was popular also. One of the students T62 concluded that it is linearly dependent because $r < n$ in R^n. This student demonstrated a misconception. This indicated that the student missed the idea. The students were simply memorising algorithms. One of the participants T53 used the ideas of geometrical interpretation to prove linear independence, but it was used wrongly. Here we can see that this supports (Gueudet-Chartier, 2004) contention that geometric concepts prevent students to develop a more formal thinking of generalised vector spaces. Participant T45 was struggling and talked about parallel vectors whilst considering the rows and columns in the matrix $\begin{bmatrix} 3 & -1 \\ 4 & 5 \\ -4 & 7 \end{bmatrix}$. From the discussion it is evident that many of the students did not develop their understanding at the process level according to APOS theory. Students find it difficult to understand concepts in linear algebra due to its abstract nature and also as a result of too many definitions and axioms that students need to comprehend, Brijlall and Ndlovu (2013). The score for item 1b are summarised in table 3.

Table 3**.** Summary of the scores for item b

Categories		**Frequency**
Wrong solutions	Unreasonable deductions made	58
	Use of the determinant method	2
Correct solution	Correct result and interpretation	13

From the table, it is evident that the students did not master the theorems leading to illogical deductions.

6. The semi structured interviews to item b

For this question, quite a number of students were interviewed. Among the interviewees student T25 and T4 were very confident and showed a good grasp of the learnt concepts such that they gave correct responses. This indicated that the students indeed were able to interiorise the actions into a process.

Below are the interview excerpts done with T23, T13, T44 and T57 when asked whether it is linearly dependent

***T23**: We say that it is not linearly dependent hmm dependent because we have three vectors here and we are using R^2.*

(Informant T23, Male, 40 years old)

***T13:** Quiet for a moment …. right we can use scalars to express one of the vectors in terms of the other or we can use the general vector to express this one, then we check for consistency at the end.*

(Informant T13, Female, 30 years old)

***T44**: These are linearly dependent, because there is no vector which is a multiple of each other and there is a certain theorem that we use that is when r is greater than n, thus all.*

(Informant T44, Male, 29 years old)

***T57**: Linearly independent because the number of vectors is greater than the number of vectors in a vector space*
***R:** In your response on the activity sheet you simply wrote since r > n, therefore they are linearly independent. What were you referring to?*
***T57**: There is this theorem where-by r stand for number of elements in a given vector and n stands for the geometric part where you say R^2 and R^3, where say R^2 it's a 2 dimension x and y, where R^3 is three dimension.*

(Informant T57, Female, 31 years old)

According to Tall (2004), the theorem was difficult to the students because it was stated in the formal way. The students struggled to apply this theorem as they confuse the unknowns and the number of equations. Hence these students could not develop their understanding at the process level.

7. Results for item e
In this item the presence of zeros was adequate for the students to reach proper conclusions. 38% of the students were able to make sound arguments and got correct answers. The results appears in table 4.

Table 4: Summary of scores for Item e

Categories		**Frequency**
Wrong solution	Use of the determinant method	8
	Incorrect theorem and unreasonable deduction	19
	Linearly dependent with unreasonable justification	18
Correct solutions	Correct theory and inference	28

Many of the students could not use inspection but calculated the determinant. Some of the students were able to identify the zero vector but made inappropriate arguments. The other students grabbed any terms that were not appropriate for example it is linearly independent because there is no solution. This caused these individuals not to have their mental construction at the process level.

8. Semi structured interviews to item e
The researcher noted that T69 came up with a 3×3 matrix and reduced it to echelon form. The following matrix was obtained after row reduction $\begin{bmatrix} 1 & -1{,}5 & 3{,}5 \\ 0 & -3{,}5 & 6{,}5 \\ 0 & 0 & 0 \end{bmatrix}$ and the student concluded that in terms of basis of R^3 and obtained $(1\ -1\frac{1}{2}\ 3\frac{1}{2})$ and $\left(0\ -3\frac{1}{2}\ \ 6\frac{1}{2}\right)$, which is incorrect. The student response shows that he was now thinking about basis of a subspace.

The following students were interviewed and these are their responses:
***R**: Can you outline whether the given vectors are linearly independent or not?*

T69: *Hmmm part e, [quiet for a moment]. Let me see there is a zero zero zero so hmmm it is not linearly independent.*
R: Give reason why it is not linearly independent?
T69: *It is linearly independent because if you do back substitution, you obtain many solutions, k_3 will be a parameter.[referring to the solution of the form $Ax = 0$.]*
R: *Be clear on the aspect on getting many solutions.*
T69: *Linearly independent hmmm [laughing]. I am now confusing myself. I am confusing the terms. If we have many solutions, it is linearly dependent because we are saying it depends on the parameter now.*

(Informant T69, Male, 36 years old)

From the analysis, student T63, had the equation $\mathrm{k}_1(2,-3,7)+\mathrm{k}_3(3,-1,4)=0$ then he concluded that there are many solutions for the given scalars. The student was probed and asked to determine linear independence, but failed to explain the necessary stages.

The interview responses indicated that the respondents did not develop their mental constructions at the process level according to APOS theory. They were still struggling to explain the concept of linear independence though some shows a better understanding after probing and were progressing towards a process conception of the concept.

Discussion

From the analysis above, **t**he study identified various shortcomings in which students struggle to find the determinant of $\mathrm{n}\times m$ matrix and making incorrect arguments after row reduction. In order to show whether a set $M_{2\times 2}$ is linearly independent, more errors were seen as the students considers determinants of separate matrices. Some students attempted to add a row of zeros forcing the matrix to be a square. Most of the students failed to base their arguments on given theorems, deviating from the demands of the question.

This demonstrates that these students failed to understand the concept of linear independence. A person who possesses conceptual knowledge is able to link pieces of information and relationships in different concepts, Hiebert (2013). This reinforces the assertion given by Ndlovu and Brijall (2015) who asserted that teachers must be aware of student's conflicting ideas and errors so as to be able to emphasise and support the learning of the new concepts they encounter.

Thomas and Stewart (2011) uphold that students fail to apply the meaning of a definition because they do not cognitively understand it, hence they are good at manipulating symbols without a deeper understanding. Bouhjar et al. (2018) also pointed out that students tend to perform better on the questions that required procedural orientation that the conceptual orientation questions. Students struggled to use theorems in the discussions as they confuse them. This causes many of these students to have their mental construction at the action level. The major limitations of the study were that it was very small with one group of first year students so that the results could not be generalised to other groups. Further studies can explore pre-service teachers' mental construction of the concept of linear independence using other frameworks.

Conclusion

This article describes how the undergraduate teachers' make the necessary mental constructions when learning linear independence using the APOS theory. The researcher came up with a genetic decomposition before teaching the concepts. It proved to be a valuable tool in assisting to identify the mental mechanisms that the participants possess.

However, I noted as a researcher that despite that the genetic decomposition was an important tool to use for conceptual development, there are a number of participants' answers that were not apprehended in the genetic decomposition. In order to put into cognisance all the students' solutions, there is the need to revise and modify it so as to accommodate them.

The researcher takes cognisance of the following schemas: (1) row reduction and interpretation of the solutions, (2) calculating determinants (3) row of zero in a matrix with particular reference to determinants. These schemas are important when developing the concept linear independence. Basing on the results, table 5 shows the revised genetic decomposition which has implication for the study. The researcher included a section on prerequisite concepts that was not captured as well as accommodating the students' responses that were not captured in the genetic decomposition.

Table 5. Initial and modified genetic decomposition

Initial Genetic Decomposition	Modified Genetic Decomposition
	The following prerequisite constructions were considered: (i) solutions after row echelon form (ii)concept of determinants
Action To prove linear independence, involves multiple step procedures, and the term acts as an external stimulus of what the individual should do. For example, equation of the form $k_1v_1 + k_2v_2 + \cdots k_nv_n = 0$. Appropriate methods are chosen to determine linear independence.	**Action** Given set S=$\{v_1, v_2, v_3\}$ to show it is linearly independent, the individual set up the equation $k_1v_1 + k_2v_2 + k_3v_3 = 0$. Solve for the scalars or come up with coefficient matrix and find determinant and make arguments whether it is linearly independent basing on the result obtained.
Process The individual here does not need to perform any of the steps, but can make arguments based on theorems. Interiorisation is reached when the individual thinks about the techniques described above without particular set of vectors.	**Process** Actions can be thought of without specific vectors. The individual can predict the nature of solutions of given set of systems of equations. and can simply apply theorems on linear independence that is theorem [1.1] and [1.2] to predict whether vectors are linearly independent/dependent without showing any working.

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